Mid Semestral Exam
Representation Theory of Finite Groups
BMath-III, 25-26

Time: 2 hours

Maximum marks: 30

Answer all questions. No marks will be awarded in absence of complete justification.

Throughout, all groups are finite. All representations are over the field of complex numbers.

1. (a) Find the character of the regular representation $\mathbb{C}G$ of a group G.
   (b) Prove that if $d_1, d_2, \ldots, d_r$ are the degrees of the distinct irreducible representations of G, then $o(G) = \sum_{i=1}^{r} d_i^2$. (2+4)

2. (a) Prove that if χ and σ are irreducible characters of groups G and H, then $\chi \times \sigma$ defined by
   $$(\chi \times \sigma)(g, h) = \chi(g)\sigma(h), \quad \forall g \in G, h \in H$$
   is also an irreducible character of $G \times H$.
   (b) Prove that every irreducible character of $G \times H$ is of the form $\chi \times \sigma$ for some irreducible characters χ and σ of G and H respectively. (2+4)

3. Compute the character table of the symmetric group S_4. (6)

4. (a) Prove that the characters of a group are algebraic integers.
   (b) Show that all characters of the symmetric group S_n are integers, for all $n \geq 2$.
   (c) Are all characters of the alternating groups $A_n, n \geq 3$, integers as well? (2+8+2)

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